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LETTER TO THE EDITOR

The kink: real Hopf bundle and Morse theory

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Abstract. We study the $(\phi^4)_2$ theory with negative mass term and consider small variations to extremals (Morse theory), identifying the Jacobi fields (zero modes). The associated topological information is recovered, interpreting the vacuum and the kink as sections in the trivial and Hopf real line bundles respectively; we consider also duality, PS limit, spinor character and statistics. We conclude with the relevance of the two other Hopf line bundles (complex and quaternionic) for vortices and instantons, respectively.

In the $(\phi^4)_2$ theory with the 'wrong' mass term, the static solutions extremise the functional

$$A=\int_{-\infty}^{\infty}\mathrm{d}x\left[\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2+\frac{\lambda}{4}(\phi^2-c^2)^2\right] \tag{1}$$

with $c^2=\mu^2/\lambda$ and μ the 'wrong' mass; the bounded solutions of $\delta A/\delta\phi=0$ are

$$\phi=\pm c,\qquad \phi=0,\qquad \phi_{\text{kink}}=\pm\frac{\mu}{\sqrt{\lambda}}\tanh\frac{\mu}{\sqrt{2}}(x-x_0), \tag{2}$$

$$\phi=M\,\mathrm{sn}(k,x-x_0); \tag{2$'$}$$

$\phi=\pm c$ is the doubly degenerate vacuum, $\phi=0$ is unstable, the third equation is the kink solution (Rajaraman 1975) centred at x_0, and (2$'$) is a real sine-amplitude elliptic solution, periodic in x space, with k and x_0 integration constants; for $k=1$ we recover the kink. There are other, elliptic, solutions, in which ϕ does not remain bounded as $x\to\pm\infty$; of course, the whole problem is equivalent to the motion $x(t)$ of a particle with $m=1$ in the one-dimensional potential $V(x)=-\frac{1}{4}\lambda\ (x^2-c^2)^2$.

The stability of these extrema are tested by the spectrum of the Hessian operator; according to Morse theory (Milnor 1963), this gives information on the topology of the manifold of solutions ; if $\bar\phi=\phi-\phi_0$, with ϕ_0 the solution of (1),

$$-\mathrm{d}^2\bar\phi/\mathrm{d}x^2-V''[\phi_0(x)]\bar\phi=-\omega^2\bar\phi \tag{3}$$

is the Hessian eigenvalue equation; the solutions (Rajaraman 1975) are

$$\begin{gathered}\phi=\pm c \text{ is a minimum},\qquad \phi=0 \text{ is a maximum},\\ \phi_{\text{kink}} \text{ is a minimum with two Jacobi (zero) modes.}\end{gathered} \tag{4}$$

One zero mode of the kink is $\bar\phi=\dot\phi$, i.e. the translation mode (going from kink centred at x_0 to kink centred elsewhere); the other zero mode leads to elliptic solution (2$'$) (in fact, the kink is a limiting sine-amplitude function). Similarly, one expects the $\mathrm{sn}(k,x)$ wavefunction to have the translational mode and a k-changing mode as zero modes.

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With these facts about the topology of the $(\phi^4)_2$ theory we turn now to a more direct topological interpretation of the solutions.

The fibre bundle interpretation of the kink is straightforward; as base space we take $R=\{x\}$ 'completed' at infinity, i.e. $R\cup\{\infty\}=S^1$; the structure group is the internal symmetry of (1), namely $Z_2(\phi\to-\phi)$. As $\Pi_0(Z_2)=Z_2$ there are just the trivial bundle $P_0=S^1\times Z_2$ and another, non-trivial bundle (G-bundles over the sphere S^n are classified by $\Pi_{n-1}(G)$, see Steenrod 1951). This bundle is very well known: it is the Hopf first bundle $S^0=Z_2\to S^1\to S^1=RP^1$, with the antipodal action of Z_2 on S^1.

We take the natural representation $Z_2\mapsto R$ to form the associated real line bundle

$$\begin{array}{llcl} \alpha: & Z_2=S^0\to & S^1\to & S^1=RP^1 \\ & \qquad\quad\downarrow & & \| \\ \xi_\alpha: & \qquad\quad R\to & E\to & S^1 \end{array} \tag{5}$$

E is just the (infinite) Möbius band, and ξ_α is the basic bundle for real vector bundles (based on the orthogonal groups, see Milnor (1974)).

Incidentally, the solitons of a broken-symmetry theory are classified by $\Pi_{d-2}(G/H)$, with d the total space–time dimension, G the symmetry group and H the stabiliser of the vacuum (see Boya *et al* 1978); here again, we have $\Pi_{2-2}(Z_2/e)=Z_2$, i.e. *the classes of bundles coincide with the classes of 'kinks'*: the vacuum class corresponds to the trivial bundle P_0, and we associate the kink with the Hopf bundle α.

In $\Gamma(\xi_\alpha)$, the space of cross sections of ξ_α, we take an arbitrary element ϕ; because ξ_α is 'twisted', we need at least two charts in the circle S^1 to describe ϕ as an ordinary function; if the second chart is 'at infinity', we just get the condition $\phi(+\infty)$ (i.e. ϕ in $\theta=\pi$ in one chart) $=-\phi(-\infty)$ (i.e., ϕ in $\theta=-\pi$ in the other), because $\phi\to-\phi$ in the overlap. This is of course what the kink function does: *the kink solution is just a smooth cross section in the vector bundle of the first Hopf sphere bundle* (also called the real Hopf line bundle).

As there is no other non-trivial bundle, there is no other kink, and therefore *there is no antikink*, which is of course well known: the Z_2 symmetry makes us work modulo two. The vacuum sector can of course be realised as a section in P_0.

One can even suggest two 'natural' sections in ξ_α which will reproduce two important limiting cases of the kink of (2).

(i) Describe the base S^1 by an angle θ, and take $\phi(\theta)=c\theta/\pi$; viewing it back in R, we reproduce a 'Prasad-Sommerfield' limit (PS) of the kink, namely $\lambda\to0$, $\mu\to0$ but $\mu/\sqrt{\lambda}=c$ fixed.

(ii) Now take $\phi(\theta=0)=c$ and 'translate' this vector parallel to the base curve (lift of d/dx vector field by the *unique* flat connection): the graph of the section would look like an ϵ(sign)-function, with a jump of $2c$ at $\theta=0$: this is the 'string limit' (strong limit) of the kink $\lambda\to\infty$, $\mu\to\infty$, $\mu/\sqrt{\lambda}=c$ constant:

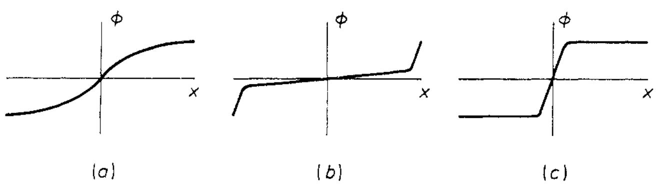

Figure 1. (*a*) General kink; (*b*) PS limit; (*c*) string limit.

The transition $(c)\to(b)\to(a)$ can be seen as the lift of a conformal transformation with generator $(1-x^2)\,\mathrm{d}/\mathrm{d}x$, see Mateos (1980).

As a further study of (1) let us write it in the form

$$2A[\phi]=\|D\|^2+\|B\|^2 \tag{6}$$

with $D=\mathrm{d}\phi/\mathrm{d}x$, $B=(\lambda/2)^{1/2}\,(\phi^2-c^2)$ and $\|\cdot\|$ an L^2 norm. One sees at once that $D=0$, $B=0$ gives the absolute minimum, namely $\phi=\pm c$; in order to identify the kink in this way (Mateos 1980) we write again

$$2A[\phi]=\|D\|^2+\|B\|^2=\|D\pm B\|^2\mp|DB| \tag{7}$$

and, as $\|D\pm B\|\geqslant 0$, we have the inequality $\|D\|^2+\|B\|^2\geqslant|DB|$; this bound is saturated for $A\pm B=0$ or

$$\mathrm{d}\phi/\mathrm{d}x=\pm(\lambda/2)^{1/2}\,(\phi^2-c^2) \tag{8}$$

with general solution $\phi_{\text{kink}}=\pm(\mu/\surd\lambda)\tanh[\mu(x-x_0)/\surd 2]$, i.e. the kink. *The kink solution saturates the* (*Bogomolny*) *bound* (see Bogomolny 1976). Of course, (8) is a first-order equation, with just an integration constant; the associated Jacobi equation is

$$\mathrm{d}\delta\phi/\mathrm{d}x=\mp(\lambda/2)^{1/2}\phi_K\delta\phi \tag{9}$$

which is satisfied by the function $\dot\phi_{\text{kink}}=\delta\phi$: *the translational mode is now the unique zero mode.*

For any scalar field in $1+1$, $\epsilon_{\mu\nu}\partial_\nu\phi=j_\mu$ is automatically conserved, and hence the associated charge, $\phi(+\infty)-\phi(-\infty)$, is called the topological charge. Here, however, it is *not* quantised, but is just $2c$. What is an integer here, characterising the bundle, is the first *Stiefel–Whitney class* w_1 of the bundle (by definition 1 in the Hirzebruch approach, see Milnor (1974)). Here we can recover it by a 'Poincaré–Hopf theorem': for $\phi\in\Gamma(\xi_\alpha)$

$$\sum_{p_i\text{ zero}} w(p_i)=w_1(\xi_\alpha)=1 \qquad (\textstyle\sum\text{ is mod }2) \tag{10}$$

where p_i are the (isolated) zeros of section ϕ, and $w(p_i)=0$ or 1 (mod 2) according to whether the graph of ϕ close to p_i does not or does cross $\phi=0$; e.g. for the kink of (3), there is just a zero, at $x=0$, of *windungzahl* $w=1$. Physically this says that, with solitons, the Higgs fields have zeros.

We observe that the dimension of the null space of normalisable Jacobi modes is 0 for the vacuum (and one for the kink): the SW class of vacuum is obviously $w_1(P_0)=e$, and as the kink bundle is a sort of square root of the vacuum bundle (see below), we recover the characteristic number by the kernel of an operator: a sort of Atiyah–Singer theorem in odd dimensions.

Perhaps a less known feature of the kink is its *spinor character.* This can be seen as follows: the 'square' of the Hopf α-bundle is just the (trivial) tangent bundle of the circle (w_1 class $=1+1$ mod $2=0$). Now the 'square root' of this tangent bundle is the spinor bundle of the circle: in general

$$\begin{array}{l} Z_2 \\ \downarrow \\ \mathrm{Pin}(n) \\ \downarrow \\ \mathrm{O}(n)\to\tau(\mathcal{V})\to\mathcal{V} \end{array} \qquad\qquad \text{lifts to }\mathrm{Pin}(n)\to\tilde\tau\,(\mathcal{V})\to\mathcal{V}$$

iff $w_2(\mathcal{V})=0$; here $\mathrm{Pin}(n)$ is the anti-image of the orthogonal groups in the Clifford algebra (Bourbaki 1960) and $\tau(\mathscr{V})$ is the bundle of orthogonal frames to a Riemann manifold $\mathscr{V}$. In our case $\mathscr{V}=S^1$, $\mathrm{O}(n)=\mathrm{O}(1)=\mathrm{Z}_2$, $\mathrm{Pin}(1)=\mathrm{Z}_4$ and $w_2(S^1)=0$. So we have $\mathrm{Pin}(1)=\mathrm{Z}_4\to\tilde{\tau}(S^1)\to S^1$, which reduces to

$$\mathrm{Spin}(1)=\mathrm{Z}_2\to S^1\to S^1=RP^1$$

because $\mathrm{O}\to\mathrm{SO}$ wherever the manifold is orientable. The written bundle is isomorphic to the Hopf α-bundle: *the kink, as section in a spinor bundle, has the character of a spinor field.*

This spinor interpretation of the kink reinforces the considerations of Finkelstein (Finkelstein and Rubinstein 1968): he proved long ago the antisymmetry under exchange of the two-kink wavefunction: as we have shown here the spinorial character of the kink field, we have an example of topological proof of the spin-statistics theorem in $1+1$ dimensions (in this case the antisymmetry is obvious in the two-kink configuration).

We have done this extensive study of the kink as the simplest soliton; we anticipate that the vortex of Nielsen–Olesen (1973) realises a section in the second Hopf bundle (or complex line bundle), and most of the aspects described here persist, with obvious modifications, with some new features emerging (e.g. a complex structure). The same bundle is connected with the magnetic monopole (see e.g. Ryder (1980) or Boya, unpublished). We shall publish our results for the vortex elsewhere.

Finally, the third Hopf sphere bundle is realised in the instanton, as should be clear from the analysis of Atiyah (Atiyah *et al* 1977). We therefore have the scheme

$$\begin{array}{llcccccl}
 & & \mathrm{U}(1)=\mathrm{SO}(2) & & & & & \\
 & & \| & & & & & \\
\text{kink} & \mathrm{Z}_2=\mathrm{O}(1)=S^0\to & S^1 & \to & S^1=RP^1 & & & \\
 & & \| & & & & & \\
\text{vortex or monopole} & \mathrm{SO}(2)=\mathrm{U}(1)= & S^1 & \to & S^3 & \to & S^2=CP^1 & \\
 & & & & \| & & & \\
\text{instanton} & \mathrm{SU}(2)=\mathrm{Sp}(1)= & & & S^3 & \to & S^7 & \to S^4=HP^1 \\
 & & & & & & \| & \\
\text{not a group; unit octonions}= & & & & & & S^7 & \to S^{15}\to S^8=\theta P^1
\end{array}$$

Is there any physics related to the fourth, octonionic line Hopf sphere bundle?

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